A van der Corput-type algorithm for LS-sequences of points

Ingrid Carbone[1]

Abstract

In this paper we associate to any *LS-sequence of partitions* $\{\rho^n_{L,S}\}$ the corresponding *LS-sequence of points* $\{\xi^n_{L,S}\}$ obtained reordering the points of each partition with an explicit algorithm. The procedure begins with the representation in base $L+S$ of natural numbers, $[n]_{L+S}$, and ends with the *LS-radical inverse function* $\phi_{L,S}$, introduced ad hoc, evaluated at an appropriate subsequence of natural numbers depending on L and S. This construction is deeply related to the geometric representation of the points of $\{\xi^n_{L,S}\}$ by suitable affine functions and reminds the van der Corput sequences in base b.

Keywords Uniform distribution, sequences of partitions, van der Corput sequences, discrepancy.

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1 Introduction

The search for new classes of *uniformly distributed* sequences of points on $[0,1[$ or in higher dimension is one of the most interesting aspects of the theory of uniform distribution, especially after the discovery that *low discrepancy* sequences can be applied to Quasi Monte-Carlo Methods. Some of them are naturally associated to sequences of partitions. The most important one is the *van der Corput sequence*, introduced in [16], which is a low discrepancy sequence of points obtained by choosing in the "right" order the left endpoints of the binary sequence of partitions $\{\delta_n\}$, where for each $n \in \mathbb{N}$

$$\delta_n = \left\{ \left[\frac{i}{2^n}, \frac{i+1}{2^n}\right[, 0 \leq i \leq 2^{n-1} \right\}.$$

We will come back to van der Corput sequence very soon.

In a recent paper [2] the author has introduced a new countable family of uniformly distributed sequences of partitions, called *LS-sequences of partitions* and

[1]University of Calabria, Department of Mathematics, Ponte P. Bucci Cubo 30B, 87036 Arcavacata di Rende (Cosenza) Italy. E-mail:i-carbone@unical.it

denoted by $\{\rho^n_{L,S}\}$; in the simple case $L = S = 1$ we get one of the *Kakutani sequences of partitions*; we have called it *Kakutani-Fibonacci sequence of partitions* and we will describe it in a very deatailed way in Section 3 and Section 4. Upper and lower estimates of the discrepancy of $\{\rho^n_{L,S}\}$ have been also given and it has been showed that $\{\rho^n_{L,S}\}$ has low discrepancy whenever $L \geq S$; in particular, the Kakutani-Fibonacci sequence has low discrepancy (it is, actually, the first time that a precise estimate of the discrepancy of one of the Kakutani sequences has been obtained).

By reordering the left endpoints of the intervals of each partition $\rho^n_{L,S}$ with an explicit algorithm described in the same article, it is has been possible to associate to each sequence of partition $\{\rho^n_{L,S}\}$ a sequence of points called *LS-sequence of points* and denoted by $\{\xi^n_{L,S}\}$. Upper bounds for the discrepancy of these sequences have been given, and the proof that each $\{\xi^n_{L,S}\}$ is a low discrepancy sequence when $L \geq S$, too. More details on $\{\rho^n_{L,S}\}$, $\{\xi^n_{L,S}\}$ and their discrepancy will be given in Section 2.

In Section 3 we present a new algorithm to construct *LS*-sequences of points $\{\xi^n_{L,S}\}$, which is formally independent at all on the concept of sequence of partitions: we do not need to reorder the points of each partitions, as it is shown in Theorem 3.5, and therefore it is simpler to use then the one introduced in [2]. Of course, the connection between each $\{\xi^n_{L,S}\}$ and the corresponding $\{\rho^n_{L,S}\}$ lies behind all the procedure, and this geometric interpretation represents one of the most interesting aspects of this new family of sequences.

In order to prove this result, we represent the points of each partition $\rho^n_{L,S}$ as subsequent compositions of suitable functions depending on L and S, as showed in Proposition 3.2.

We conclude this paper with a significative example.

Let us complete this Introduction with some classical definitions, starting from the constructions of the van der Corput sequence.

Any natural number n can be expressed as

$$n = \sum_{k=0}^{M} a_k(n)\, 2^k \,, \tag{1}$$

with $a_k(n) \in \{0,1\}$ for all $0 \leq k \leq M$, with $M = [\log_2 n]$ (here and in the sequel $[\cdot]$ denotes the integer part). Such representation is unique modulus the addition of higher powers of 2 with zero coefficients.

The expression (1) leads to the *binary representation*

$$[n]_2 = a_M(n) a_{M-1}(n) \dots a_0(n) \,. \tag{2}$$

The representation of n in base 2 given by (1) is used to define the *radical-inverse function* ϕ_2 on $\mathbb{N}$ which associates to the string of digits (2) the number

$$\phi_2(n) = \sum_{k=0}^{M} a_k(n) 2^{-k-1},$$

whose binary representation is $0.a_0(n)a_1(n)\ldots a_M(n)$.

Of course $0 \leq \phi_2(n) < 1$ for all $n \geq 0$ and the 2-radix notation of $\phi_2(n)$ actually is $[\phi_2(n)]_2 = 0.a_0(n)a_1(n)\ldots a_M(n)$.

The van der Corput sequence is the sequence

$$\{\phi_2(n)\}_n.$$

After its introduction in 1935, several generalizations of the van der Corput sequence have been given during the subsequent decades. Among them, we mention the so called *van der Corput sequence in base b* denoted by $\{\phi_b(n)\}_n$, where b is any positive integer, with $\phi_b(n) = \sum_{k=0}^{M} a_k(n) b^{-k-1}$ and $a_k(n) \in \{0, 1, \ldots, b-1\}$. These sequences were introduced by Hammersley ([11]) and their discrepancy was first studied by Halton ([10]), who proved that these sequences have low discrepancy for all b.

Another generalization is the so called *scrambled van der Corput sequence* denoted by $\{\phi_b^{\sigma}(n)\}_n$, where $\phi_b^{\sigma}(n) = \sum_{k=0}^{M} \sigma_k(a_k(n)) b^{-k-1}$ and $\{\sigma_n\}_n$ is a sequence of permutations of the set $\{0, 1, \ldots, b-1\}$. Faure ([8] and [9]) proved that, for some suitable $\{\sigma_n\}_n$, the discrepancy of $\{\phi_b^{\sigma}(n)\}_n$ is asymptotically better that the discrepancy of $\{\phi_b(n)\}_n$.

We need some definitions (see [7] and [14]).

Definition 1.1. Given a sequence of points $\{x_n\}$ of the interval $[0,1[$, we say that $\{x_n\}$ is *uniformly distributed (u.d.)* if for any $a, b \in \mathbb{R}$ such that $0 \leq a < b \leq 1$ we have

$$\lim_{N \to \infty} \frac{1}{N} \sum_{j=1}^{N} \chi_{[a,b[}(x_j) = b - a.$$

Definition 1.2. Given a point set $X_N = (x_1, x_2, \ldots, x_N)$ of the interval $[0,1[$, the *discrepancy* of X_N is

$$D(X_N) = \sup_{0 \leq a < b \leq 1} \left| \frac{1}{N} \sum_{j=1}^{N} \chi_{[a,b[}(x_j) - (b-a) \right|$$

and the *star-discrepancy* of X_N is

$$D^\star(X_N) = \sup_{0<b<1} \left| \frac{1}{N} \sum_{j=1}^{N} \chi_{[0,b[}(x_j) - b \right|.$$

Given a sequence of points $X = \{x_n\}$ in $[0,1[$, for each N we can consider the point set $X_N = (x_1, x_2, \ldots, x_N)$ and the associated sequences $\{D(X_N)\}_N$ (and, respectively, $\{D^*(X_N)\}_N$). It is important to study how these sequences behave when N tends to infinity.

It is well known that a sequence $X = \{x_n\}$ is u.d. if and only if the discrepancy $D(X_N)$ tends to 0 whenever N tends to infinity, and also that there exists an absolute constant c such that $c \log N \leq ND(X_N) \leq N$ (see [15]).

We say that $X = \{x_n\}$ has *low discrepancy* if there exists a constant $C > 0$ such that $ND(X_N) \leq C \log N$ for any N.

If we consider a sequence of partitions $\{\pi_n\}$ of $[0,1[$, with

$$\pi_n = \{[y_j^{(n)}, y_{j+1}^{(n)}[, 1 \leq j \leq t_n\},$$

each partition π_n can be regarded as a point set. For this reason, we can re-write the definition of uniform distribution and discrepancy, given for sequences of points, in this particular setting.

We say that the sequence of partitions $\{\pi_n\}$ is u.d. if for any real numbers a, b such that $0 \leq a < b \leq 1$ we have

$$\lim_{n\to\infty} \frac{1}{t_n} \sum_{j=1}^{t_n} \chi_{[a,b[}(y_j^{(n)}) = b - a.$$

Moreover, since each π_n is defined by t_n points, the concepts of discrepancy and star-discrepancy can be naturally extended to the sequence $\{\pi_n\}$ as follows:

$$D(\pi_n) = \sup_{0\leq a<b<1} \left| \frac{1}{t_n} \sum_{j=1}^{t_n} \chi_{[a,b[}(y_j^{(n)}) - (b-a) \right|$$

and

$$D^\star(\pi_n) = \sup_{0<b<1} \left| \frac{1}{t_n} \sum_{j=1}^{t_n} \chi_{[0,b[}(y_j^{(n)}) - b \right|.$$

We say that $\{\pi_n\}$ has low discrepancy if there exists a constant $C > 0$ such that $t_n D(\pi_n) \leq C$ for any n (see [2]).

In [17] it has been introduced a new class of u.d. sequences of partitions called ρ-*refinements*.

Definition 1.3. For any fixed non trivial finite partition ρ of $[0,1[$, the ρ-*refinement* of a partition π of $[0,1[$ (denoted by $\rho\pi$) is obtained by subdividing all the intervals of π having maximal length positively (or directly) homothetically to ρ. If for any $n \in \mathbb{N}$ we denote by $\rho^n\pi$ the ρ-refinement of $\rho^{n-1}\pi$, we get a sequences of partitions $\{\rho^n\pi\}$, called the *sequence of successive* ρ-*refinements* of π.

If $\rho = \{[0,\alpha[, [\alpha,1[\}$ for some $\alpha \in [0,1[$ and $\omega = \{[0,1[\}$ is the trivial partition, the sequence $\{\rho^n\omega\}$ is actually the well known *Kakutani's sequence* of α-refinements, which has been defined in [13] and will be denoted by $\{\kappa_n\}$. Kakutani proved that

Theorem 1.4. *The sequence* $\{\kappa_n\}$ *is uniformly distributed.*

In [17] it has been proved the following generalization:

Theorem 1.5. *The sequence* $\{\rho^n\omega\}$ *is uniformly distributed.*

The uniform distribution of the sequence $\{\rho^n\pi\}$, when π is not the trivial partition, has been investigated in [1], where nice necessary and sufficient conditions on π and ρ have been found.

A further contribution to the theory of ρ-refinement, moreover, has been given by Drmota and Infusino who, in [6], provided estimates of the discrepancy of sequences of ρ-refinements and, in particular, of *LS*-sequences of partions.

In the last years, however, the interest on Kakutani's splitting procedure gave birth to several generalizations, among which we would like to mention the extension to separable metric spaces [3] and the generalization to higher dimensions which is intrinsically *n*-dimensional [4]. Moreover, it has been extended to some fractals in [12].

In [5] the authors present a von Neumann-type theorem for sequences of partitions, showing actually that a sequence of partitions having an infinitesimal diameter can be reordered in a sequence of partitions which is uniformly distributed.

2 *LS*-sequences

In this section we recall the definition of *LS-sequence of points* $\{\xi^n_{L,S}\}$ introduced in [2]. To this aim, we need the definition of *LS-sequence of partitions* $\{\rho^n_{L,S}\}$.

Definition 2.1. Let us fix two positive integers L and S and let γ be the positive solution of $L\gamma + S\gamma^2 = 1$. Denote by $\rho_{L,S}$ the partition defined by L “long” intervals having length γ followed by S “short” intervals having length γ^2. The sequence of the successive $\rho_{L,S}$-refinements of the trivial partition ω is called *LS-sequence of partitions* and is denoted by $\{\rho^n_{L,S}\,\omega\}$ (or $\{\rho^n_{L,S}\}$ for short).

If we denote with t_n the total number of intervals of $\rho^n_{L,S}$, with l_n the number of its long intervals and with s_n the number of its short intervals, it is very simple to see that $t_n = l_n + s_n$, $l_n = L\,l_{n-1} + s_{n-1}$ and $s_n = S\,l_{n-1}$. We deduce that $t_n = L\,t_{n-1} + S\,t_{n-2}$ (with $t_0 = 1$ and $t_1 = L + S$) which has the explicit solution

$$t_n = \frac{1+S\gamma}{1+S\gamma^2}\left(\frac{1}{\gamma}\right)^n - \frac{S\gamma - S\gamma^2}{1+S\gamma^2}(-S\gamma)^n.$$

We note that l_n and s_n satisfy the same difference equation, with initial conditions $l_0 = 1$, $l_1 = L$ and $s_0 = 0$, $s_1 = S$, respectively.

In [2] we have given precise estimates from above and from below of the discrepancy of *LS*-sequences of partitions, and proved the following

Theorem 2.2. *i) If $S \le L$ there exist $c_1, c_1' > 0$ such that for any $n \in \mathbb{N}$*

$$c_1' \le t_n D(\rho^n_{L,S}) \le c_1\,.$$

ii) If $S = L + 1$ there exist $c_2, c_2' > 0$ such that for any $n \in \mathbb{N}$

$$c_2' \log t_n \le t_n D(\rho^n_{L,S}) \le c_2 \log t_n\,.$$

iii) If $S \ge L + 2$ there exist $c_3, c_3' > 0$ such that for any $n \in \mathbb{N}$

$$c_3' {t_n}^{1-\tau} \le t_n D(\rho^n_{L,S}) \le c_3\, t_n^{1-\tau}\,,$$

where $1 - \tau = -\frac{\log(S\gamma)}{\log\gamma} > 0$.

We emphasize that each *LS*-sequence of partitions with $S \le L$ has low discrepancy.

Let us now recall the definition of the sequence of points $\{\xi^n_{L,S}\}$ associated to the sequence of partitions $\{\rho^n_{L,S}\}$.

Definition 2.3. We fix a sequence of partitions $\{\rho_{L,S}^n\}$. For each $n \geq 1$ we define the two families of functions

$$\varphi_i^{(n+1)}(x) = x + i\gamma^{n+1} \quad \text{and} \quad \varphi_{L,j}^{(n+1)}(x) = x + L\gamma^{n+1} + j\gamma^{n+2} \tag{3}$$

for $1 \leq i \leq L$ and $1 \leq j \leq S-1$. We denote by $\Lambda_{L,S}^1$ the set made by the t_1 left endpoints of the intervals of $\rho_{L,S}^1$ ordered by magnitude. If $\Lambda_{L,S}^n = \left(\xi_1^{(n)}, \ldots, \xi_{t_n}^{(n)}\right)$ denotes the set of the t_n points defining $\rho_{L,S}^n$, ordered in the appropriate way, the set $\Lambda_{L,S}^{n+1}$, which consists in a reordering of the points defining $\rho_{L,S}^{n+1}$, is obtained as follows:

$$\begin{aligned}
&\Lambda_{L,S}^{n+1} = \Big(\xi_1^{(n)}, \xi_2^{(n)}, \ldots, \xi_{t_n}^{(n)},\\
&\varphi_1^{(n+1)}(\xi_1^{(n)}), \ldots, \varphi_1^{(n+1)}(\xi_{l_n}^{(n)}), \ldots, \varphi_L^{(n+1)}(\xi_1^{(n)}), \ldots, \varphi_L^{(n+1)}(\xi_{l_n}^{(n)}),\\
&\varphi_{L,1}^{(n+1)}(\xi_1^{(n)}), \ldots, \varphi_{L,1}^{(n+1)}(\xi_{l_n}^{(n)}), \ldots, \varphi_{L,S-1}^{(n+1)}(\xi_1^{(n)}), \ldots, \varphi_{L,S-1}^{(n+1)}(\xi_{l_n}^{(n)})\Big).
\end{aligned} \tag{4}$$

The sequence obtained reordering successively the points of $\rho_{L,S}^{n+1} \setminus \rho_{L,S}^n$ is called *LS-sequence of points* and is denoted by $\{\xi_{L,S}^n\}$.

In [2] we also provided the following bounds for the discrepancy of these sequences of points.

Theorem 2.4. *i) If $S \leq L$ there exists $k_1 > 0$ such that for any $N \in \mathbb{N}$*

$$N\,D\Big(\xi_{L,S}^1, \xi_{L,S}^2, \ldots, \xi_{L,S}^N\Big) \leq k_1 \log N\,.$$

ii) If $S = L+1$ there exists $k_2, c_2' > 0$ such that for any $N \in \mathbb{N}$

$$c_2' \log N \leq N\,D\Big(\xi_{L,S}^1, \xi_{L,S}^2, \ldots, \xi_{L,S}^N\Big) \leq k_2 \log^2 N\,.$$

iii) If $S \geq L+2$ there exists $k_3, c_3' > 0$ such that for any $N \in \mathbb{N}$

$$c_3' N^{1-\tau} \leq N\,D\Big(\xi_{L,S}^1, \xi_{L,S}^2, \ldots, \xi_{L,S}^N\Big) \leq k_3 N^{1-\tau} \log N\,,$$

where $1 - \tau = -\frac{\log(S\gamma)}{\log \gamma} > 0$.

Theorem 2.2 and Theorem 2.4 show that if the LS-sequence of partitions has low discrepancy, the corresponding LS-sequence of points has low discrepancy, too. Moreover, c_2' and c_3' denote the same constants in both theorem as, of course, whenever $N = t_n$ for some n, it follows that $D\left(\xi_{L,S}^1, \xi_{L,S}^2, \dots, \xi_{L,S}^N\right) = D(\rho_{L,S}^n)$.

Note that the lower bounds just follow from Theorem 2.2. In fact we believe that the discrepancy of all the LS-sequences of points coincides with the upper bound given by Theorem 2.4.

3 The new definition " à la van der Corput"

From now on we fix L, S and γ such that $L\gamma + S\gamma^2 = 1$.

The first step is to represent the points of the sequence $\{\xi_{L,S}^n\}$ writing the elements of each set $\Lambda_{L,S}^n$ as images of the point $x = 0$ under the compositions of suitable functions we are going to define.

To this purpose we introduce the functions ψ_i on restricted domains as follows: for every $0 \le i \le L-1$ we consider

$$\psi_i(x) = \gamma x + i\gamma \quad \text{restricted to} \quad 0 \le x < 1, \tag{5}$$

while for every $L \le i \le L+S-1$ we define

$$\psi_i(x) = \gamma x + L\gamma + (i-L)\gamma^2 \quad \text{restricted to} \quad 0 \le x < \gamma. \tag{6}$$

We observe that the functions (5) map $[0,1[$ onto $[i\gamma,(i+1)\gamma[$ and the functions (6) map $[0,\gamma[$ onto $[L\gamma+(i-L)\gamma^2, L\gamma+(i-L+1)\gamma^2[$. Therefore, the compositions $\psi_i \circ \psi_j$ are not defined if and only if $L \le i \le L+S-1$ and $1 \le j \le L+S-1$.

Let us denote by $E_{L,S}$ the set consisting of all the pairs of indices which correspond to the "forbidden" compositions, i.e.

$$E_{L,S} = \{L, L+1, \dots, L+S-1\} \times \{1, \dots, L+S-1\}. \tag{7}$$

In order to present our main result, we have to give the explicit expression of the compositions $\psi_{i_1 i_2 \dots i_n} = \psi_{i_1} \circ \psi_{i_2} \circ \dots \psi_{i_n}$ of the functions (5) and (6).

Lemma 3.1. *For any $n \in \mathbb{N}$ and any n-tuple $(i_1 i_2 \dots i_n)$ of elements of the set $\{0, 1, \dots, L+S-1\}$ such that $(i_h, i_{h+1}) \notin E_{L,S}$ for any $1 \le h \le n-1$, we have*

$$\psi_{i_1 i_2 \dots i_n}(x) = \gamma^n x + \sum_{k=1}^{n} b_k \gamma^k, \tag{8}$$

where $b_k = i_k$ if $0 \le i_k \le L-1$, while $b_k = L + (i_k - L)\gamma$ if $L \le i_k \le L+S-1$.

Proof. We prove (8) by induction on n.

If $n = 2$, because of (5) and (6) we have only three kinds of compositions ψ_{i_1,i_2}:

a) if $0 \le i_1, i_2 \le L-1$, we have $\psi_{i_1,i_2}(x) = \gamma^2 x + \sum_{k=1}^{2} b_k \gamma^k$, where $b_k = i_k$;

b) if $0 \le i_1 \le L-1$ and $L \le i_2 \le L+S-1$, a simple calculation gives (8), with $b_1 = i_1$ and $b_2 = L + (i_2 - L)\gamma$;

c) if $L \le i_i \le L+S-1$ and $0 \le i_2 \le L-1$, we obtain the same expression, where $b_1 = L + (i_1 - L)\gamma$ and $b_2 = i_2$.

Suppose now that (11) holds for $n \ge 1$ and let us prove it for $n+1$, namely that

$$\psi_{i_1 i_2 \dots i_n i_{n+1}}(x) = \gamma^{n+1} x + \sum_{k=1}^{n+1} b_k \gamma^k \tag{9}$$

for all $i_{n+1} \ne 0$. Of course, at this stage we have to distinguish between the two kinds of functions (5) and (6).

If $0 \le i_{n+1} \le L-1$, we simply obtain (9) with $b_{n+1} = i_{n+1}$.

If $L \le i_{n+1} \le L+S-1$, simple calculations give again (9), with $b_{n+1} = L + (i_{n+1} - L)\gamma$. So the induction is complete. □

An immediate consequence of the previous lemma is the following result.

Proposition 3.2. *The first t_n points of the LS-sequence $\{\xi^n_{L,S}\}$ are*

$$\Lambda^n_{L,S} = \{\psi_{i_1 i_2 \dots i_n}(0)\}, \tag{10}$$

where $(i_h, i_{h+1}) \notin E_{L,S}$ for any $1 \le h \le n-1$ and $(i_n, i_{n-1}, \dots, i_1)$ are the n-tuples of elements of the set $\{0, 1, \dots, L+S-1\}$, ordered with respect to the magnitude of the numbers $i_n i_{n-1} \dots i_1$ in base $L+S$.

Proof. Taking (5) and (6) into account, we see that

$$\Lambda^1_{L,S} = \{\psi_0(0), \psi_1(0), \dots, \psi_{L+S-1}(0)\},$$

which are exactly the $L+S$ left endpoints of the intervals of $\rho^1_{L,S}$ ordered by magnitude, so (10) is true for $n = 1$.

Fix any integer $n \ge 1$ and suppose (10) is true for n.

If $i_{n+1} = 0$, as $\psi_0(0) = 0$ we have

$$\psi_{i_1 i_2 \dots i_n i_{n+1}}(0) = \psi_{i_1 i_2 \dots i_n}(0),$$

and we simply observe that this way we get all the points of $\Lambda_{L,S}^n$. All the new points of $\Lambda_{L,S}^{n+1} \setminus \Lambda_{L,S}^n$ are obtained when $i_{n+1} \neq 0$ as follows.

If $1 \leq i_{n+1} \leq L$ we have

$$\begin{aligned}
\psi_{i_1 i_2 \dots i_n i_{n+1}}(0) &= i_{n+1}\gamma^{n+1} + \sum_{k=1}^{n} b_k \gamma^k = \psi_{i_1 i_2 \dots i_n}(0) + i_{n+1}\gamma^{n+1} \\
&= \varphi_{i_{n+1}}^{(n+1)}(\psi_{i_1 i_2 \dots i_n}(0)),
\end{aligned}$$

according to (3).

If $L+1 \leq i_{n+1} \leq L+S-1$, we set $j_{n+1} = i_{n+1} - L$, with $1 \leq j_{n+1} \leq S-1$, and write

$$\begin{aligned}
&\psi_{i_1 i_2 \dots i_n i_{n+1}}(0) = (L + j_{n+1}\gamma)\gamma^{n+1} + \sum_{k=1}^{n} b_k \gamma^k \\
= \; &\psi_{i_1 i_2 \dots i_n}(0) + L\gamma^{n+1} + j_{n+1}\gamma^{n+2} = \varphi_{L, j_{n+1}}^{(n+1)}(\psi_{i_1 i_2 \dots i_n}(0)).
\end{aligned}$$

At this point we recall that the set $\Lambda_{L,S}^{n+1} \setminus \Lambda_{L,S}^n$ is made by $(L+S-1)l_n$ points as we evaluate the $L+S-1$ functions defined by (3) at the first l_n points of $\Lambda_{L,S}^n$ (see (4)).

It remains only to prove that the total number of the points of $\Lambda_{L,S}^{n+1} \setminus \Lambda_{L,S}^n$ we add in the way described above are exactly $(L+S-1)l_n$. We denote by $d_{L,S}^{(n)}$ the total number of n-tuples containing two consecutive digits of the set $E_{L,S}$ defined by (7). Of course, we have $d_{L,S}^{(n)} = (L+S)^n - t_n$.

We observe, in fact, that the points of $\Lambda_{L,S}^{n+1} \setminus \Lambda_{L,S}^n$ come from $L+S-1$ blocks of all the $(n+1)$-tuples of elements of the set $\{0, 1, \dots, L+S-1\}$, which are $(L+S)^n$, ordered by magnitude of the numbers in base $L+S$. Moreover, the total number of not admissible $(n+1)$-tuples which are contained in these blocks is given by the difference between $(L+S-1)(L+S)^n$ and the total number of forbidden ones, which is exactly $d_{L,S}^{(n+1)} - d_{L,S}^{(n)}$. In other words,

$$\begin{aligned}
&(L+S-1)(L+S)^n - \left(d_{L,S}^{(n+1)} - d_{L,S}^{(n)}\right) \\
&= (L+S-1)(L+S)^n - \left((L+S)^{n+1} - t_{n+1} - (L+S-1)^n + t_n\right) \\
&= t_{n+1} - t_n = (L+S-1)l_n.
\end{aligned}$$

The theorem is completely proved. $\square$

We are now ready to present LS-sequences of points from another point of view, which is deeply connected to the explicit geometric representations given in Proposition 3.2. More precisely, we want to show how to get the points of these sequences starting from the representation in base $L+S$ of natural numbers. This algorithm is very simple and allows us to compute directly the points of each $\{\xi^n_{L,S}\}$.

Given
$$n = \sum_{k=0}^{M} a_k(n)(L+S)^k,$$
where $a_k(n) \in \{0,1,2,\ldots,L+S-1\}$ for all $0 \le k \le M$ and $M = [\log_{L+S} n]$, its representation in base $L+S$ is
$$[n]_{L+S} = a_M(n)\, a_{M-1}(n) \ldots a_0(n)\,.$$

If we reverse the order of the digits of $[n]_{L+S}$ and add a dot, we get
$$[\phi_{L+S}(n)]_{L+S} = 0.a_0(n)\, a_1(n) \ldots a_{M-1}(n)\, a_M(n)\,,$$
which is the representation in base $L+S$ of the $(L+S)$-radical inverse function ϕ_{L+S} defined on $\mathbb{N}$ by
$$\phi_{L+S}(n) = \sum_{k=0}^{M} a_k(n)(L+S)^{-k-1}.$$

Definition 3.3. We denote by $\mathbb{N}_{L,S}$ the set of all positive integers n, ordered by magnitude, with $[n]_{L+S} = a_M(n)\, a_{M-1}(n) \ldots a_0(n)$ such that $(a_k(n), a_{k+1}(n)) \notin E_{L,S}$ for all $0 \le k \le M-1$.

Definition 3.4. For all $n \in \mathbb{N}_{L,S}$ we define the *LS-radical inverse function* as follows:
$$\phi_{L,S}(n) = \sum_{k=0}^{M} \tilde{a}_k(n)\, \gamma^{k+1}\,, \tag{11}$$
where $\tilde{a}_k(n) = a_k(n)$ if $0 \le a_k(n) \le L-1$ and $\tilde{a}_k(n) = L + \gamma(a_k(n) - L)$ if $L \le a_k(n) \le L+S-1$.

Taking the above definitions into account, we are able to present the main result of this paper.

Theorem 3.5. *Any LS-sequence of points coincides with the sequence* $\{\phi_{L,S}(n)\}$ *defined on* $\mathbb{N}_{L,S}$.

Proof. Fix $n \in \mathbb{N}_{L,S}$. Taking Lemma 3.1 and Definition 3.4 into account, we write

$$\phi_{L,S}(n) = \psi_{n_0 n_1 \ldots n_M}(0),$$

which proves the theorem. $\square$

Example 3.6. We consider the Kakutani-Fibonacci sequence of points $\{\xi_{1,1}^n\}$ corresponding to $L = S = 1$ and $\gamma = \frac{1}{2}(\sqrt{5} - 1)$. See Fig. 1.

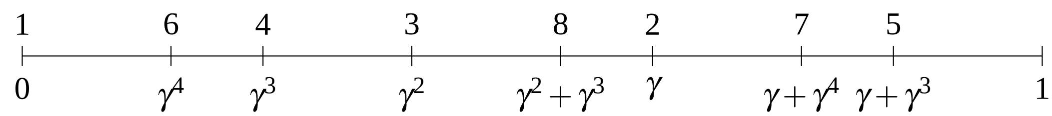

Figure 1: The first 8 points of $\{\xi_{1,1}^n\}$.

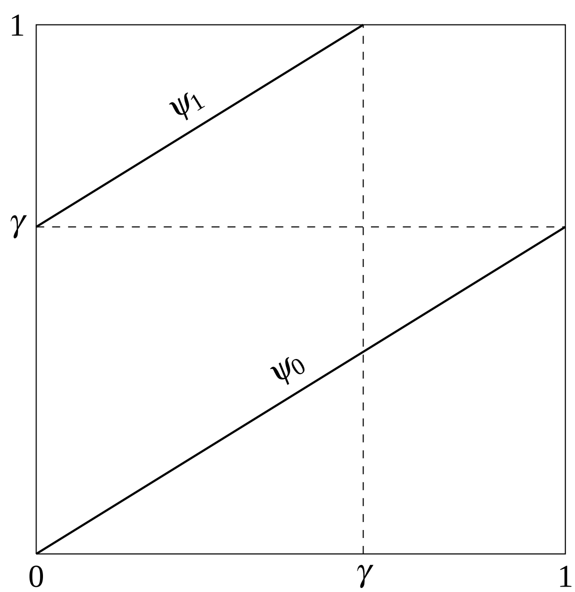

Figure 2: The functions ψ_0 and ψ_1 associated to $\{\xi_{1,1}^n\}$

The functions defined by (8) and (6) reduce to $\psi_0(x) = \gamma x$ for $0 \le x < 1$ and $\psi_1(x) = \gamma x + \gamma$ for $0 \le x < \gamma$. See the Fig. 2.

According to Definition 3.4, $\mathbb{N}_{1,1}$ is the set of all natural numbers n such that the binary representation (2) does not contain two consecutive digits equal to 1. Moreover, the $(1,1)$-radical inverse function defined by (11) on $\mathbb{N}_{1,1}$ is

$$\phi_{1,1}(n) = \sum_{k=0}^{M} a_k(n)\,\gamma^{k+1}\,.$$

By Theorem 3.5, the Kakutani-Fibonacci sequence of points $\{\xi_{1,1}^n\}$ coincides with the sequence $\{\phi_{1,1}(n)\}$.

The following table shows the construction of the first 12 points of $\{\xi_{1,1}^n\}$, where the first column contains the first 12 elements of $\mathbb{N}_{1,1}$.

$$\begin{array}{lccclccccr}
0 \to & [0]_2 & = & 0 & \to & 0.0 & = & [\phi_{1,1}(0)]_2 & \to & 0 & = \xi_{1,1}^{1} \\
1 \to & [1]_2 & = & 1 & \to & 0.1 & = & [\phi_{1,1}(1)]_2 & \to & \gamma & = \xi_{1,1}^{2} \\
2 \to & [2]_2 & = & 10 & \to & 0.01 & = & [\phi_{1,1}(2)]_2 & \to & \gamma^2 & = \xi_{1,1}^{3} \\
4 \to & [4]_2 & = & 100 & \to & 0.001 & = & [\phi_{1,1}(4)]_2 & \to & \gamma^3 & = \xi_{1,1}^{4} \\
5 \to & [5]_2 & = & 101 & \to & 0.101 & = & [\phi_{1,1}(5)]_2 & \to & \gamma+\gamma^3 & = \xi_{1,1}^{5} \\
8 \to & [8]_2 & = & 1000 & \to & 0.0001 & = & [\phi_{1,1}(8)]_2 & \to & \gamma^4 & = \xi_{1,1}^{6} \\
9 \to & [9]_2 & = & 1001 & \to & 0.1001 & = & [\phi_{1,1}(9)]_2 & \to & \gamma+\gamma^4 & = \xi_{1,1}^{7} \\
10 \to & [10]_2 & = & 1010 & \to & 0.0101 & = & [\phi_{1,1}(10)]_2 & \to & \gamma^2+\gamma^4 & = \xi_{1,1}^{8} \\
16 \to & [16]_2 & = & 10000 & \to & 0.00001 & = & [\phi_{1,1}(16)]_2 & \to & \gamma^5 & = \xi_{1,1}^{9} \\
17 \to & [17]_2 & = & 10001 & \to & 0.10001 & = & [\phi_{1,1}(17)]_2 & \to & \gamma+\gamma^5 & = \xi_{1,1}^{10} \\
18 \to & [18]_2 & = & 10010 & \to & 0.01001 & = & [\phi_{1,1}(18)]_2 & \to & \gamma^2+\gamma^5 & = \xi_{1,1}^{11} \\
20 \to & [20]_2 & = & 10100 & \to & 0.00101 & = & [\phi_{1,1}(20)]_2 & \to & \gamma^3+\gamma^5 & = \xi_{1,1}^{12}
\end{array}$$

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